Spline Method for Nonlinear Optimal Thrust Vector Controls for Atmospheric Interceptor Guidance

H. Schneider* and P. B. Reddy†
Dynamics Research Corporation, Wilmington, Mass.

Theme

THE solution of a general nth-order nonlinear system of differential equations is analytically approximated by an nth-degree spline polynomial series. This spline approach is applied to determine the optimal thrust vector controls for guiding an interceptor to a target in dense atmosphere. The problem formulation utilizes the calculus of variations leading to a Two Point Boundary Value (TPBV) problem. The spline series, in conjunction with a one- or multidimensional parameter search procedure, provides a novel optimization technique for transforming the optimality state and adjoint nonlinear differential equations with specified boundary conditions into algebraic solution expressions that are easily calculated. Boundary conditions at the initial and terminal times are also satisfied algebraically on each sweep (iteration). A noniterative guidance law is obtained as an approximate solution to the TPBV problem. Two-dimensional atmospheric intercepts are considered with fixed terminal time (fixed altitude of intercept), zero terminal miss distance, and a quadratic performance index consisting of the integral of the square of the thrust, which is a measure of fuel consumption. The quick convergence of the technique to the true optimal TPBV solution is illustrated with computer results under the severe conditions of a poor starting nominal, rapidly varying components of thrust, and large atmospheric drag nonlinearities.

Content

The development of an accurate guidance law for performing real-time atmospheric intercepts of re-entering bodies as viewed by a radar is a complex problem that utilizes the combined disciplines of nonlinear estimation, system identification, and optimal control. Nonlinear estimation and system identification are involved in estimating the target and interceptor states and unknown aerodynamic parameters from a given time sequence of discrete noisy radar observations. These observations are filtered or smoothed in real time to obtain the target initial condition and aerodynamic parameters that enable prediction of its future trajectory.

Once the target has been identified as being threatening, optimal control techniques are involved in guiding an interceptor to perform the kill. The primary interceptor control for this purpose is assumed to be the time-varying vector thrust, $U(t)$. Interceptor launch occurs at $t=0$ in order to hit the target at a fixed terminal time (t_f) corresponding to a specific desired altitude of intercept, y_f. The predicted target coordinates at t_f, namely $x_f=0, y_f$ given, are therefore random variables whose values will change with each new set of radar measurements filtered subsequent to interceptor launch.

Presented as Paper 74-827 at the AIAA Mechanics and Control of Flight Conference, Anaheim, Calif., Aug. 5-9, 1974; submitted Aug. 13, 1974; revision received April 5, 1976; synoptic received Oct. 28, 1976. Full revised paper available from National Technical Information Service, Springfield, Va., 22151, as N77-12789 at the standard price (available upon request).

Index categories: Navigation, Control, and Guidance Theory; Entry Vehicle Dynamics and Control.

*Senior Systems Analyst. Member AIAA.
†Senior Systems Analyst.

Because large unmodelled systematic sources of error exist during re-entry that preclude accurate aim point prediction for more than a few seconds, the interceptor command guidance law sought must be capable of responding quickly to large perturbations in the aim point and parameters of the model. The optimization technique described here is believed to have this capability. Figure 1 shows the two-dimensional problem addressed.

Classical methods for solving TPBV problems are known to be generally very time consuming and may not converge for problems with severe nonlinearities, when a good starting solution or "nominal" is not available.‡ Splines have powerful approximating and function representation properties that make them very useful for applications.‡ A spline series solution for the vector nth-order nonlinear differential equation

$$\overset{n}{r}(t) = f(t, r, \dot{r}, \ldots, \overset{n-1}{r}) \tag{1}$$

can be written as

$$r(t) = \sum_{m=0}^{n-1} \overset{m}{r}_0 \frac{(t-T_1)^m}{m!} + \sum_{k=1}^{j} C_{ik}\, \overset{n}{r}_k \tag{2a}$$

$$(T_j \le t \le T_{j+1};\ j=1, J)$$

$$\overset{m}{r}(t) = \frac{\partial^m}{\partial t^m} r(t) \qquad (m=0, 1, \ldots; n-1) \tag{2a$'$}$$

$$C_{ik} \equiv \begin{cases} \sum_{m=1}^{n} \Phi_{m,k} t_i^{n-m} & (k<j) \\ \dfrac{T_{ij}^n}{n!} & (k=j) \\ 0 & (k>j) \end{cases} \tag{2b}$$

$$\Phi_{m,k} \equiv \frac{(-1)^{m+1}(T_{k+1}^m - T_k^m)}{m!(n-m)!} \qquad T_{ik} \equiv t_i - T_k \tag{2c}$$

‡References are presented in the full paper.

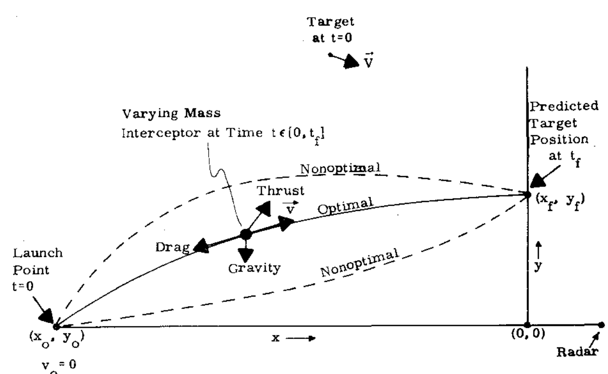

Fig. 1 Two-dimensional intercept problem.

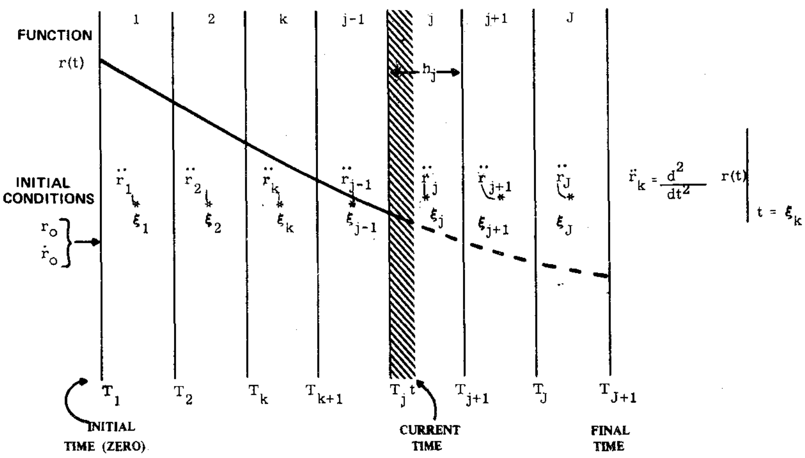

Fig. 2 Spline regions and parameters ($n=2$).

where the C_{ik} are given time-dependent weighting (or integration) coefficients multiplying the nth-order state time derivatives ($\overset{n}{r}_k$) evaluated at the spline region centers (ξ_k-points of Fig. 2).

Equation (2a), containing $(n+j)$ terms, is an nth-degree spline polynomial series of much utility. The latter summation approximates *all* of the remainder terms of the Taylor series. The spline approximation error to $\overset{m}{r}(t)$, $m<n$, shrinks to zero as the spline intervals $h_k \to 0$ in Fig. 2. The first $(n-1)$ derivatives of $r(t)$, given by Eq. (2a′), are continuous everywhere. The $\overset{n}{r}_k$ are constant over each spline region with a jump discontinuity occurring at the "knots" or T_k points, $k=2, J$. Each $\overset{n}{r}_k$ parameter controls the "curvature" of $r(t)$ in the kth spline region so that local variations in $r(t)$ are easily followed, thus reflecting the known good function representation and approximating properties of splines. The initial condition parameters, $\overset{m}{r}_0$, $m<n$, and $\overset{n}{r}_k$, occur linearly.[§]

The h_k can be chosen arbitrarily small so that Eq. (1) can be satisfied over a step size corresponding to any a priori numerical integration step size. Since Eq. (2a) with $\overset{n}{r}_k = f_k$ already satisfies Eq. (1), no accompanying numerical integration routines such as Runga Kutta are needed and all major computations are performed only at the ξ_k points. In practice, it is often found that the h_k can be chosen "large" for good accuracy.

In view of the above, Eqs. (2) are suitable for application to nonlinear estimation and TPBV optimal control problems.

For the thrust vector control problem, the point-mass dynamic equations of a variable-mass interceptor are

$$\ddot{r} = M^{-1}[-(D+\dot{M})\dot{r} + U] - g_0 \qquad g_0^T = [0, 32.2] \tag{3a}$$

$$\dddot{r} = M^{-1}\{-\dot{M}(2\ddot{r}+g_0) - [\ddot{r} + \dot{r}(v^{-2}\dot{r}^T\ddot{r} - \frac{\dot{y}}{H_s})]D + cr\dot{\mathrm{U}} + \dot{U}\} \tag{3a′}$$

$$\dot{M} = -c\mathrm{U} \tag{3b}$$

$$\ddot{M} = -c\dot{\mathrm{U}} \tag{3b′}$$

where

$$r^T \equiv [x(t), y(t)] \qquad U^T \equiv [U_x(t), U_y(t)] \qquad \mathrm{U} \equiv |U|$$

$$D(r, \dot{r}) \equiv (\rho_s/2\beta)\, v\, e^{-y/H_s} \qquad v \equiv (\dot{x}^2 + \dot{y}^2)^{1/2}$$

$$M(t) \equiv m(t)/m(t_0) \qquad 0 < M_{\min} \le M(t) \le 1$$

β is the ballistic coefficient (psf) assumed constant, and ρ_s (lb/ft^3) is the sea-level air density. A flat earth, exponential atmosphere with scale height $H_s \approx 22{,}000$ ft, and constant gravity g_0, have also been assumed. The D-terms are the x, y components of the drag acceleration (ft/sec^2) and U_x, U_y are the x, y components of $U(t)$. v(ft/sec) is the total speed and $M(t)$ is the mass ratio with t_0 (zero) being the interceptor launch time. A quadratic performance index for fuel expenditure is

$$\Lambda = \int_0^{t_f} (U_x^2 + U_y^2)\,\mathrm{d}t \tag{4}$$

The necessary conditions for optimality via the Calculus of Variations and Pontryagin Minimum Principle, in state variable notation with $x^T \equiv [x, y, \dot{x}, \dot{y}, M]$, are

$$\dot{x} = f(x, U, t) \qquad \text{(5 state differential equations)} \tag{5a}$$

$$\dot{p} = -\left(\frac{\partial \mathcal{H}}{\partial x}\right) \qquad \text{(5 adjoint differential equations)} \tag{5b}$$

$$\frac{\partial \mathcal{H}}{\partial U} = 0 \qquad \text{(2 algebraic control equations)} \tag{5c}$$

where the variational Hamiltonian, $\mathcal{H}$, is given by

$$\mathcal{H} = U_x^2 + U_y^2 + p^1\dot{x} + p^2\dot{y} + p^3\ddot{x} + p^4\ddot{y} - p^5 c\mathrm{U} \tag{5d}$$

The right-hand side of Eq. (3a) is substituted for $\ddot{x}$, $\ddot{y}$.

The boundary conditions are

$$r(0) = r_0 \qquad \dot{r}(0) = \dot{r}_0 \qquad M(0) = 1 \qquad r(t_f) = r_f$$

$$p^3(t_f) = p^4(t_f) = p^5(t_f) = 0 \qquad \text{(all are given)} \tag{6}$$

implying no thrusting at the terminal time ($U_f = 0$). Equations (4-6) form a difficult nonlinear TPBVP. Equations (5a) are highly nonlinear functions of both state and control variables. Although Eqs. (5b) are always linear in the adjoint variables by virtue of the definition of the Hamiltonian, the coefficients of the p's are nonlinear functions of x and U. Further, U occurs nonlinearly in Eq. (5c) which gives an implicit algebraic relationship between U and the state and adjoint variables. An explicit dependence is always obtainable via a quasilinearization. The full paper describes the spline optimization technique as applied to Eqs. (4-6). Cubic, quadratic, and "linear" splines corresponding to $n=3, 2, 1$ are used for $r(t)$, $M(t)$, and $p(t)$, respectively. The $\overset{n}{r}_k$ derivatives in the spline series are computed from the dynamic Eqs. (3a′, 3b′, 5b), respectively. Lower order derivatives are computed from the *spline series*. Algebraic relationships are obtained for all initial and terminal condition parameter values that reflect both the given dynamical constraints and required boundary conditions. A one- or multidimensional parameter search procedure is utilized for improving and accelerating convergence. Computational results are presented for some special cases of interest.

The approach is analytical, flexible, and more generally applicable to problems in optimal control and estimation.

[§]Replacement of n by $(n+1)$ in Eq. (2a) with r_k^{n+1} approximated by the first difference $[\overset{n}{r}(T_{k+1}) - \overset{n}{r}(T_k)]/h_k$ leads to an alternative useful series (omitted) with a different definition of C_{ik} with Eq. (1) satisfied at the knots instead of the ξ_k points. Recursive formulas for the direct numerical integration of Eq. (1) can be derived by algebraic manipulation.